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Trivially noncontractible edges in a contraction critically 5-connected graph

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Abstract

An edge of a k-connected graph is said to be k-contractible if the contraction of the edge results in a k-connected graph. A k-connected graph with no k-contractible edge is said to be contraction critically k-connected. An edge of a k-connected graph is said to be trivially noncontractible if its end vertices have a common neighbor of degree k. We prove that a contraction critically 5-connected graph on n vertices has at least $n/2$ trivially noncontractible edges and at least $(2n)/9$ vertices of degree 5.

MSC: 05C40

Keywords: 5-connected graph; Contractible edge; Contraction critically k-connected

1. Introduction

In this paper, we deal with finite undirected graphs with neither loops nor multiple edges. For a graph G, let $V(G)$ and $E(G)$ denote the set of vertices of G and the set of edges of G, respectively. For a vertex $x \in V(G)$, we denote by $N_G(x)$ the neighborhood of x in G and let $N_G[x] = N_G(x) \cup \{x\}$. Moreover, for a subset $S \subset V(G)$, let $N_G(S) = \bigcup_{x \in S} N_G(x) - S$. We denote the degree of $x \in V(G)$ by $d_G(x)$, namely $d_G(x) = |N_G(x)|$. We denote the minimum degree of G by $\delta(G)$. We denote by K_4^- the graph obtained from K_4 by deleting one edge. The *square* of a graph G is the graph obtained from G by adding edges joining each

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pair of vertices whose distance in G is 2. Let G be a connected graph. A subset $S \subset V(G)$ is said to be a *cutset* of G, if $G - S$ is not connected. A cutset S is said to be a *k-cutset* if $|S| = k$. The neighborhood of a vertex of degree k of a k-connected graph is called a *trivial* cutset.

Let k be an integer such that $k \geqslant 2$ and let G be a k-connected graph. An edge e of G is said to be *k-contractible* if the contraction of the edge results in a k-connected graph. If an edge is not k-contractible, then it is called a *noncontractible* edge. If the contraction of $e \in E(G)$ results in a graph with minimum degree $k - 1$, then e is said to be *trivially* noncontractible. In other words, e is trivially noncontractible if and only if the end vertices of e have a common neighbor of degree k, or (equivalently) they are contained in some trivial cutset. If G does not have a k-contractible edge, then G is said to be *contraction critically k-connected*.

It is known that every 3-connected graph of order 5 or more contains a 3-contractible edge (Tutte [10]).

The characterization of contraction critically 4-connected graphs was obtained by Fontet [4] and independently by Martinov [8]. Namely they proved the following theorem.

Theorem A. *If G is a* 4*-connected graph with no* 4*-contractible edge, then G is either the square of a cycle or the line graph of a cyclically* 4*-connected* 3*-regular graph.*

From Theorem A, we know that each edge of a contraction critically 4-connected graph is trivially noncontractible.

Thomassen [9] proved that each k-connected triangle-free graph has a k-contractible edge. Thomassen also stated that, for $k \geqslant 4$, there exist infinitely many k-connected k-regular graphs each of whose edge is trivially noncontractible.

W. Mader [7] proved the following theorem which states that each contraction critically k-connected graph has many triangles.

Theorem B. *Let G be a k-connected graph of order n with no contractible edges. Then G contains at least $n/3$ triangles.*

There is a contraction critically 5-connected graph which is not 5-regular. However, from Egawa's result [3] and Kriesell's result [5] we know that the minimum degree of a contraction critically 5-connected graph is 5. Ando et al. [2] investigated conditions for minimally k-connected graphs to have a contractible edge, moreover, Ando et al. [1] proved the following theorem which says that each contraction critically 5-connected graph has many vertices of degree 5.

Theorem C. *Let G be a* 5*-connected graph on n vertices which does not have a* 5*-contractible edge. Then each vertex of G has a neighbor of degree* 5 *and G has at least $n/5$ vertices of degree* 5*.*

From Theorems B and C, it seems to be a natural expectation that each contraction critically 5-connected graph has many trivially noncontractible edges. In this paper, we consider the distribution of trivially noncontractible edges in a contraction critically

5-connected graph. The knowledge of their distribution brings us an improvement of Theorem C. Our main results are the following.

Theorem 1. *Each contraction critically* 5*-connected graph of order n has at least* $n/2$ *trivially noncontractible edges.*

Theorem 2. *Each contraction critically* 5*-connected graph of order n has at least* $(2n)/9$ *vertices of degree* 5.

The organization of the paper is as follows. Section 2 contains preliminary results. We give a key proposition in Section 3. We prove Theorems 1 and 2 in Section 4.

2. Preliminaries

In this section we give some more definitions and preliminary results.

For a graph G, we write $|G|$ for $|V(G)|$. For subgraphs A and B of a graph G, when there is no ambiguity, we write simply A for $V(A)$ and B for $V(B)$. So $N_G(A)$ and $A \cap B$ mean $N_G(V(A))$ and $V(A) \cap V(B)$, respectively. Also for a subgraph A of G and a subset S of $V(G)$ we write $A \cap S$ and $A \cup S$ for $V(A) \cap S$ and $V(A) \cup S$, respectively. For $S \subseteq V(G)$, we let $G[S]$ denote the subgraph induced by S in G, and let $G - S$ denote the graph obtained from G by deleting the vertices in S together with the edges incident with them; thus $G - S = G[V(G) - S]$. Let $V_k(G)$ denote the set of vertices of degree k. For a vertex x in $V(G)$, let $E(x)$ denote the set of edges incident with x. When there is no ambiguity, we write $E(S)$ for $E(G[S])$. For subsets S and T of $V(G)$, we denote by $E_G(S, T)$ the set of edges between S and T. If $S = \{x\}$, then we simply write $E_G(x, T)$ instead of $E_G(\{x\}, T)$.

A subgraph A of a k-connected graph G is called a *fragment* if $|N_G(A)| = k$ and $V(G) - (A \cup N_G(A)) \neq \emptyset$. In other words, a fragment A is a nonempty union of components of $G - S$ where S is a k-cutset of G such that $V(G) - (A \cup S) \neq \emptyset$. By the definition if A is a fragment of G, then $G - (A \cup N_G(A))$ is also a fragment of G.

Let A be a fragment of a k-connected graph G and let e be an edge of G. Then A is said to be a fragment *with respect to e* if $V(e) \subset N_G(A)$. For a set of edges $F \subset E(G)$, we say that A is a fragment *with respect to F* if A is a fragment with respect to some $e \in F$. A fragment A with respect to F is said to be *minimum* (resp. *minimal*) if there is no fragment B other than A with respect to F such that $|B| < |A|$ (resp. $B \subset A$). A fragment A is said to be *trivial* if $|A| = 1$. Let e be an edge of G which is not k-contractible. Then there is a k-cutset S such that $e \in E(S)$. We denote the cardinality of a minimum fragment with respect to e by $\eta(e)$ and we set $E^{(i)}(G) = \{e \in E(G) \mid \eta(e) = i\}$. Moreover, we set $E_L(G) = \{e \in E(G) \mid \eta(e) \geqslant \lceil (k+1)/2 \rceil\}$. By the definition, $e \in E^{(1)}(G)$ if and only if e is contained in some trivial cutset. Thus $E^{(1)}(G)$ is the set of trivially noncontractible edges of G. Note that if G is a contraction critically 5-connected graph, then $E(G) = E^{(1)}(G) \cup E^{(2)}(G) \cup E_L(G)$ since $\lceil (k+1)/2 \rceil = 3$.

The following is an immediate observation.

Lemma 2.1. *Let S be a k-cutset of a k-connected graph G and let* $x \in S$. *If there is a vertex y of G such that* $N_G[y] \supset N_G(x) - S$, *then* $y \in S$.

Proof. Let A be a fragment of $G-S$ and let $\bar{A}=G-(S\cup A)$. Assume that $y\notin S$. Without loss of generality we may assume that $y\in A$. Then we observe $N_G[y]\subset(S\cup A)$. Since $N_G[y]\supset N_G(x)-S$ we have $N_G(x)\subset(S\cup A)$, that is $N_G(x)\cap\bar{A}=\emptyset$ which contradicts the choice of S. Now Lemma 2.1 is proved. $\square$

The following Lemma states some elementary facts which play essential roles in our arguments.

Lemma 2.2. *Let G be a k-connected graph, and let S and T be k-cutsets of G. Let A and B be fragments of $G-S$ and $G-T$, respectively. Let $\bar{A}=G-(S\cup A)$ and $\bar{B}=G-(T\cup B)$.*

B	$\bar{A}\cap B$	$S\cap B$	$A\cap B$
T	$\bar{A}\cap T$	$S\cap T$	$A\cap T$
$\bar{B}$	$\bar{A}\cap\bar{B}$	$S\cap\bar{B}$	$A\cap\bar{B}$
	$\bar{A}$	S	A

Then the following hold:

(a) *If $|(S\cap B)\cup(S\cap T)\cup(A\cap T)|>k$, then $|(\bar{A}\cap T)\cup(S\cap T)\cup(S\cap\bar{B})|<k$ and $\bar{A}\cap\bar{B}=\emptyset$.*
(b) *If $A\cap B\neq\emptyset$, then $|S\cap B|\geqslant|\bar{A}\cap T|$.*

Proof. (a) Since S and T are both k-cutsets,

$$|S|+|T|=|(S\cap B)\cup(S\cap T)\cup(S\cap\bar{B})|+|(\bar{A}\cap T)\cup(S\cap T)\cup(A\cap T)|=2k.$$

Hence, if the hypothesis of (a) holds, then $|(\bar{A}\cap T)\cup(S\cap T)\cup(S\cap\bar{B})|<k$ and this implies that $\bar{A}\cap\bar{B}=\emptyset$ since G is k-connected.

(b) Since neither $\bar{A}$ nor $\bar{B}$ is empty, $A\cap B\neq\emptyset$ means that $(S\cap B)\cup(S\cap T)\cup(A\cap T)$ is a cutset of G. Hence $|(S\cap B)\cup(S\cap T)\cup(A\cap T)|\geqslant k$ since G is k-connected. From this inequality and $k=|T|=|(\bar{A}\cap T)\cup(S\cap T)\cup(A\cap T)|$, we have $|S\cap B|\geqslant|\bar{A}\cap T|$. $\square$.

The following Lemma 2.3 due to Mader [6,7] is fundamental.

Lemma 2.3. *Let G be a k-connected graph. Let F be a nonempty subset of $E_L(G)$. If there is a minimal fragment with respect to F which has a vertex x with $E(x)\cap F\neq\emptyset$, then G has a k-contractible edge.* $\square$

The following is an immediate consequence of Lemma 2.3.

Corollary 2.4. *Let G be a k-connected graph. If there is a vertex $x\in V(G)$ such that $E(x)\subset E_L(G)$, then G has a k-contractible edge.* $\square$

The following is an easy but useful observation.

Lemma 2.5. *Let G be a k-connected graph and let x be a vertex of G. Let A be a minimum fragment with respect to an edge in $E(x)$. Let xu be a noncontractible edge in $E_G(x,A)$ and T be a k-cutset which contains x and u. In this situation, if $\eta(xu)\geqslant|A|$, then $A\subset T$.*

Proof. Let $N_G(A) = S$ and $\bar{A} = G - (S \cup A)$. Let B be a fragment of $G - T$ and let $\bar{B} = G - (T \cup B)$. Note that $x \in S \cap T$ and $u \in A \cap T$.

Claim 2.5.1. *If* $\eta(xu) \geqslant |A| - 1$, *then either* $A \cap B$ *or* $A \cap \bar{B}$ *is empty*.

Proof. Note that $x \in S \cap T$ and $u \in A \cap T$. By way of contradiction, assume that $\eta(xu) \geqslant |A| - 1$ and neither $A \cap B$ nor $A \cap \bar{B}$ is empty. Then we observe that

$$|A \cap B| \leqslant |A| - |A \cap T| - |A \cap \bar{B}| \leqslant |A| - |\{u\}| - |A \cap \bar{B}| \leqslant |A| - 2.$$

By symmetry, we have $|A \cap \bar{B}| \leqslant |A| - 2$. Since $\eta(xu) \geqslant |A| - 1$ we know that neither $A \cap B$ nor $A \cap \bar{B}$ is a fragment with respect to *xu*. Hence, since $xu \in E_G(x, A)$, we observe that both $|(S \cap B) \cup (S \cap T) \cup (A \cap T)|$ and $|(S \cap \bar{B}) \cup (S \cap T) \cup (A \cap T)|$ are greater than *k*. By Lemma 2.2(a), this implies that $|\bar{A} \cap T| < |A \cap T|$ and $\bar{A} \cap B = \bar{A} \cap \bar{B} = \emptyset$ which means that $\bar{A} = \bar{A} \cap T$. Then $|\bar{A}| = |\bar{A} \cap T| < |A \cap T| < |A|$ which contradicts the choice of *A*. Now Claim 2.5.1 is proved. □

Now we prove Lemma 2.5. Suppose that $T \not\supset A$. Then, by Claim 2.5.1, without loss of generality we may assume that $A \cap B \neq \emptyset$ and $A \cap \bar{B} = \emptyset$. Since

$$|A \cap B| \leqslant |A| - |A \cap T| \leqslant |A| - 1$$

and $|A| \leqslant \eta(xu)$ we know that $A \cap B$ is not a fragment with respect to *xu*. Hence, since $xu \in E_G(x, A)$, we observe that $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| > k$. Then Lemma 2.2(a) assures us that $\bar{A} \cap \bar{B} = \emptyset$ which implies that $\bar{B} = S \cap \bar{B}$ since $A \cap \bar{B} = \emptyset$. However, since $A \cap B \neq \emptyset$, Lemma 2.2(b) tells us that $|S \cap B| \geqslant |\bar{A} \cap T|$, which (since $|S| = |T|$) is the same as $|S \cap \bar{B}| \leqslant |A \cap T|$. Thus $|\bar{B}| = |S \cap \bar{B}| \leqslant |A \cap T| < |A| \leqslant \eta(xu)$ which contradicts the definition of $\eta(xu)$. Now Lemma 2.5 is proved. □

Lemma 2.6. *Let G be a* 5-*connected graph. Let* $x \in V(G)$ *be a vertex of degree* 5 *and let* $N_G(x) = \{u_1, u_2, v_1, v_2, w\}$. *If either* (a) *or* (b) *holds, then G has a contractible edge.*

(a) $G[\{u_1, u_2, v_1, v_2\}] \cong K_4$,
(b) $G[\{u_1, u_2, v_1, v_2\}] \cong K_4 - u_1v_1$, $N_G[u_1] \cup \{v_1\} \supset N_G(u_2)$ *and* $wv_1 \notin E(G)$, *where* $K_4 - u_1v_1$ *stands for the graph obtained from* K_4 *by deleting the edge* u_1v_1.

Proof. Assume that *G* has no contractible edge. Let *S* be a 5-cutset which contains both *x* and *w*. Let *A* be a fragment of $G - S$ and let $\bar{A} = G - S \cup A$.

Suppose $G[\{u_1, u_2, v_1, v_2\}] \cong K_4$. Then, without loss of generality, we may assume that $\{u_1, u_2, v_1, v_2\} \cap A = \emptyset$. Since $w \in S$, this implies that $N_G(x) \cap A = \emptyset$ contradicting the choice of *A*. Now it is shown that if (a) holds then *G* has a contractible edge.

Next suppose that (b) holds. Since neither $N_G(x) \cap A$ nor $N_G(x) \cap \bar{A}$ is empty, we observe that $u_2, v_2 \in S$ and $\{u_1, v_1\} \subset A \cup \bar{A}$. Without loss of generality we may assume that $u_1 \in \bar{A}$ and $v_1 \in A$. In this situation we know that $N_G(x) \cap A = \{v_1\}$. Also the condition $N_G[u_1] \cup \{v_1\} \supset N_G(u_2)$ assures us that $N_G(u_2) \cap A = \{v_1\}$. Furthermore, the last condition of (b), $wv_1 \notin E(G)$, tells us that $A - \{v_1\} \neq \emptyset$ since $w \in S$. Hence we observe

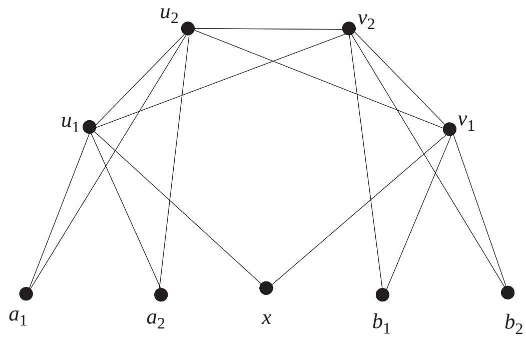

Fig. 1. A K_4^--configuration with center x.

that $(S - \{x, u_2\}) \cup \{v_1\}$ is a 4-cutset of G which contradicts the fact that G is 5-connected. Now it is shown that if (b) holds then G has a contractible edge and Lemma 2.6 is proved. $\square$

3. Key proposition

In this section we prove a proposition which plays a key role in this paper. First we introduce two definitions.

Let $S = \{a_1, a_2, x, b_1, b_2\}$ be a 5-cutset of a 5-connected graph G and let A be a component of $G - S$ such that $V(A) \subset V_5(G)$, $|V(A)| = 4$ and $G[A] \cong K_4^-$, say $A = \{u_1, u_2, v_1, v_2\}$, with edges within A and between A and S exactly as in Fig. 1; there may also be edges between vertices of S. We call this configuration, $G[V(A) \cup S]$, a K_4^--*configuration with center* x.

Next we define the notion of orthogonal edges. Two edges xu, xv of a 5-connected graph are said to be *mutually orthogonal* if there are minimum fragments B and A with respect to xu and xv, respectively, such that $u \in A$ and $v \in B$.

The following is the key proposition.

Proposition 3.1. *Let x be a vertex of a contraction critically* 5-*connected graph G. If there is no trivially noncontractible edge in $E(x)$, then G has a K_4^--configuration with center x.*

The following lemma plays an essential role in the proof of Proposition 3.1.

Lemma 3.2. *Let x be a vertex of a contraction critically* 5-*connected graph G. If there are mutually orthogonal edges in $E(x) \cap E^{(2)}(G)$, then G has a K_4^--configuration with center x.*

Proof. Let xu_1 and xv_1 be mutually orthogonal edges in $E(x) \cap E^{(2)}(G)$. Let B and A be minimum fragments with respect to xu_1 and xv_1, respectively, such that $u_1 \in A$ and $v_1 \in B$. Say $A = \{u_1, u_2\}$ and $B = \{v_1, v_2\}$. Let $S = N_G(A)$ and $\bar{A} = G - (S \cup A)$. Let $T = N_G(B)$ and $\bar{B} = G - (T \cup B)$.

Since $|A| = \eta(xv_1)$ and $|B| = \eta(xu_1)$, by Lemma 2.5, we know that $T \supset A$ and $S \supset B$, so that $|A \cap T| = |S \cap B| = 2$ and $\bar{A} \cap B = A \cap B = A \cap \bar{B} = \emptyset$.

Claim 3.2.1. $S \cap T = \{x\}$.

Proof. By way of contradiction assume that $|S \cap T| \geqslant 2$. Then

$$|(S \cap B) \cup (S \cap T) \cup (A \cap T)| = 2 + |S \cap T| + 2 \geqslant 6.$$

Hence, by Lemma 2.2(a), we have $\bar{A} \cap \bar{B} = \emptyset$ which implies that $\bar{B} = \bar{B} \cap S$. Thus $|\bar{B}| = |\bar{B} \cap S| = |S| - |S \cap T| - |S \cap B| \leqslant 5 - 2 - 2 = 1$ which contradicts the assumption that $xu_1 \in E^{(2)}(G)$. Now Claim 3.2.1 is proved. $\square$

By Claim 3.2.1 we know that $|S \cap \bar{B}| = |\bar{A} \cap T| = 2$, say $S \cap \bar{B} = \{a_1, a_2\}$ and $\bar{A} \cap T = \{b_1, b_2\}$, so that $S = \{x, a_1, a_2, v_1, v_2\}$ and $T = \{x, b_1, b_2, u_1, u_2\}$. Let $A' = \{u_1, u_2, v_1, v_2\}$. Then by Claim 3.2.1 we also know that A' is a fragment of G with which the 5-cutset $S' = N_G(A') = \{a_1, a_2, x, b_1, b_2\}$ is associated. Let $\bar{A}' = G - (S' \cup A')$.

Claim 3.2.2. $N_G(u_2) \supset (S - \{x\})$ *and* $N_G(v_2) \supset (T - \{x\})$.

Proof. We show that $N_G(u_2) \supset (S - \{x\})$. If $d_G(u_2) = 6$, then $N_G(u_2) = S \cup \{u_1\}$. So we assume that $d_G(u_2) = 5$. If $xu_2 \in E(G)$, then $N_G(u_2) \supset \{x, u_1\}$ which means that $\{u_2\}$ is a fragment with respect to xu_1. This contradicts the fact that $\eta(xu_1) = 2$. Thus $xu_2 \notin E(G)$ and $N_G(u_2) \supset (S - \{x\})$. By the same argument, $N_G(v_2) \supset (T - \{x\})$. Now Claim 3.2.2 is proved. $\square$

By Claim 3.2.2 we know that $G[A'] = G[u_1, u_2, v_1, v_2]$ contains all possible edges except possibly u_1v_1; in particular $u_2v_2 \in E(G)$. Let C be a fragment with respect to u_2v_2. Let $R = N_G(C)$ and let $\bar{C} = G - (R \cup C)$.

Claim 3.2.3. *If* $u_1v_1 \in E(G)$, *then* $u_1v_1 \in E(R)$.

Proof. Assume that $u_1v_1 \in E(G)$. Then $N_G(u_1) \supset \{x, v_1\}$ which implies that $d_G(u_1) \geqslant 6$ since $\eta(xv_1) = 2$. Since $d_G(u_1) \geqslant 6$ we know that

$$N_G[u_1] = \{u_1, u_2, v_1, v_2, x, a_1, a_2\} \supset N_G(u_2).$$

Hence, by Lemma 2.1, $u_1 \in R$. By the same argument, $v_1 \in R$. Now Claim 3.2.3 is proved. $\square$

Claim 3.2.4. $u_1v_1 \notin E(G)$.

Proof. By way of contradiction, assume that $u_1v_1 \in E(G)$. Then $G[A'] \cong K_4$. Without loss of generality, we may assume that $|S' \cap \bar{C}| \geqslant |S' \cap C|$. Then, we know that $|S' \cap C| \leqslant 2$. By Claim 3.2.3, we know that $A' \subset R$, which implies that $A' \cap C = \emptyset$. We show that $S' \cap C \neq \emptyset$. Since $A' \cap C = \emptyset$ and $C \neq \emptyset$ either $S' \cap C \neq \emptyset$ or $\bar{A}' \cap C \neq \emptyset$. If $\bar{A}' \cap C \neq \emptyset$, then

$|R-A'\cap R|+|S'\cap C|=|(\bar{A}'\cap R)\cup(S'\cap R)|+|S'\cap C|\geqslant 5$ which implies that $S'\cap C\neq\emptyset$ since $|R-A'\cap R|\leqslant 1$. Hence we observe that $S'\cap C\neq\emptyset$.

We show that $(S'\cap C)\cap\{a_1,a_2,b_1,b_2\}=\emptyset$. By way of contradiction assume that $(S'\cap C)\cap\{a_1,a_2,b_1,b_2\}\neq\emptyset$. Without loss of generality we may assume that $a_1\in(S'\cap C)$. Then, since $|A'|=|A'\cap R|=4>2\geqslant|S'\cap C|$, by Lemma 2.2(b), we know that $\bar{A}'\cap C=\emptyset$. Thus, since $N_G(a_1)\subset C\cup R$ and $A'\subset R$, we observe that $N_G(a_1)\subset((S'\cap C)-\{a_1\})\cup(R-\{v_1,v_2\})$ because $N_G(a_1)\cap\{v_1,v_2\}=\emptyset$. This means that $|N_G(a_1)|\leqslant|((S'\cap C)-\{a_1\})|+|R-\{v_1,v_2\}|\leqslant(2-1)+3=4$ which contradicts the fact that G is 5-connected. Now it is shown that $(S'\cap C)\cap\{a_1,a_2,b_1,b_2\}=\emptyset$ which means that $S'\cap C=\{x\}$ since $S'=\{x,a_1,a_2,b_1,b_2\}$ and $S'\cap C\neq\emptyset$.

Since $4=|A'\cap R|>|S'\cap C|=1$, Lemma 2.2(b) tells us that $\bar{A}'\cap C=\emptyset$ and thus $C=\{x\}$. Now we know that $x\in V_5(G)$ and $N_G(x)=R$, so that $G[N_G(x)]\supset G[A']\cong K_4$. Then by Lemma 2.6(a), G has a contractible edge which contradicts the assumption that G is contraction critical. Now Claim 3.2.4 is proved. □

By Claim 3.2.4 we know that $G[A']\cong K_4-u_1v_1$. Let R' be a 5-cutset which contains u_1 and v_2. Let C' be a fragment of $G-R'$ and let $\bar{C}'=G-(R'\cup C')$.

Claim 3.2.5. *If $xu_2\in E(G)$, then $R'\supset A'$.*

Proof. Since $xu_2\in E(G)$, $N_G[u_2]=\{u_1,u_2,v_1,v_2,a_1,a_2,x\}\supset N_G(u_1)$, hence by Lemma 2.1 we know that $u_2\in R'$. Since $u_1\in R'$,

$$N_G[v_1]=\{u_2,v_1,v_2,b_1,b_2,x\}\supset N_G(v_2)-R',$$

hence again by Lemma 2.1 we know that $v_1\in R'$. Now it is shown that $R'\supset A'$. □

The following Claim is the final step of the proof of Lemma 3.2.

Claim 3.2.6. *$xu_2,xv_2\notin E(G)$.*

Proof. We show that $xu_2\notin E(G)$. By a similar argument we can show that $xv_2\notin E(G)$. By way of contradiction assume that $xu_2\in E(G)$. Then by Claim 3.2.5, we know that $R'\subset A'$, say $R'=A'\cup\{w\}$.

We show that $\bar{A}'\cap R'=\{w\}$. Assume on the contrary that $\bar{A}'\cap R'\neq\{w\}$ which means that $\bar{A}'\cap R'=\emptyset$. Since $A'\subset R'$ we know that $A'\cap C'=A'\cap\bar{C}'=\emptyset$ which implies that neither $S'\cap C'$ nor $S'\cap\bar{C}'$ is empty, since otherwise $R'-A'$ would be a cutset, contradicting the 5-connectedness of G. Then both $|S'\cap(C'\cup R')|$ and $|S'\cap(\bar{C}'\cup R')|$are at most 4. This together with the fact that $\bar{A}'\cap R'=\emptyset$ assures us that $\bar{A}'\cap C'=\bar{A}'\cap\bar{C}'=\emptyset$ and hence $\bar{A}'=\emptyset$, a contradiction. Now it is shown that $\bar{A}'\cap R'=\{w\}$; in particular $wv_1\notin E(G)$.

Without loss of generality, we may assume that $|S'\cap\bar{C}'|\geqslant|S'\cap C'|$. Then by the same argument as in the proof of Claim 3.2.4, we can show that $C'=\{x\}$ and $N_G(x)=R'$. Now we observe that $x\in V_5$, $N_G(x)\supset A'=\{u_1,u_2,v_1,v_2\}$, $G[A']\cong K_4-u_1v_1$, $N_G[u_1]\cup\{v_1\}\supset N_G(u_2)$ and $wv_1\notin E(G)$. By Lemma 2.6(b), we conclude that G has a contractible edge which contradicts the assumption that G is contraction critically. Now Claim 3.2.6 is proved. □

By Claims 3.2.2, 3.2.4 and 3.2.6, we know that $\{u_1, u_2, v_1, v_2\} \cup \{a_1, a_2, x, b_1, b_2\}$ forms a K_4^--configuration and the proof of Lemma 3.2 is completed. $\square$

Lemma 3.3. *Let x be a vertex of a contraction critically* 5*-connected graph G. If* $E(x) \cap E^{(1)}(G) = \emptyset$ *and there is a minimum fragment A with respect to* $E(x) \cap E^{(2)}(G)$ *such that* $E_G(x, A) \cap E^{(2)}(G) \neq \emptyset$*, then G has a* K_4^-*-configuration with center x.*

Proof. Let $S = N_G(A)$ and let $\bar{A} = G - (S \cup A)$. Let *xu* be an edge in $E_G(x, A) \cap E^{(2)}(G)$ and let *B* be a minimum fragment with respect to *xu*. Let $T = N_G(B)$ and let $\bar{B} = G - (T \cup B)$. We show that $B \subset S$. By Lemma 2.5, we know that $A \subset T$ which implies that $A \cap B = \emptyset$. If $\bar{A} \cap B \neq \emptyset$ then, by Lemma 2.2(b), $|S \cap B| \geqslant |A \cap T| = |A| = 2$, which is impossible since $|S \cap B| = |B| - |\bar{A} \cap B| \leqslant 1$. Now it is shown that $\bar{A} \cap B = \emptyset$ and hence $B \subset S$. Thus there is an edge $xv \in E_G(x, B) \cap E(S)$ such that $\eta(xv) \leqslant 2$ since $xv \in E(S)$. Since $E(x) \cap E^{(1)}(G) = \emptyset$ we know that $xv \in E^{(2)}(G)$. Thus *xu* and *xv* are mutually orthogonal edges in $E(x) \cap E^{(2)}(G)$. Now the conclusion of Lemma 3.3 follows from Lemma 3.2. $\square$

Proof of Proposition 3.1. Let *G* be a contraction critically 5-connected graph. Let *x* be a vertex of *G* such that $E(x) \cap E^{(1)}(G) = \emptyset$. Then since $E(x) = E(x) \cap (E^{(1)}(G) \cup E^{(2)}(G) \cup E_L(G))$, we know that $E(x) = E(x) \cap (E^{(2)}(G) \cup E_L(G))$. If $E(x) \cap E_L(G) = \emptyset$, then there is a minimum fragment *A* with respect to $E(x) \cap E^{(2)}(G)$ such that $E_G(x, A) \cap E^{(2)}(G) \neq \emptyset$; thus the conclusion of Proposition 3.1 follows from Lemma 3.3. Hence we assume that $E(x) \cap E_L(G) \neq \emptyset$. Let *A* be a minimum fragment with respect to $E(x) \cap E_L(G)$, and take an edge $xy \in E(x) \cap E_L(G)$ such that *A* is a fragment with respect to *xy*. Let $xu \in E_G(x, A)$. Let $S = N_G(A)$ and let $\bar{A} = G - (S \cup A)$. Since *G* has no 5-contractible edge, it follows from Lemma 2.3 that $xu \notin E_L(G)$, which means that $xu \in E^{(2)}(G)$. Let *B* be a minimum fragment with respect to *xu*. Let $T = N_G(B)$ and let $\bar{B} = G - (T \cup B)$.

Claim 3.1.1. $\bar{A} \cap B = \emptyset$.

Proof. By way of contradiction assume that $\bar{A} \cap B \neq \emptyset$. Then, since $|B| = 2$ we know that $|S \cap B| \leqslant |B| - |\bar{A} \cap B| \leqslant 1$. On the other hand since $\bar{A} \cap B \neq \emptyset$, by Lemma 2.2(b), we have $|S \cap B| \geqslant |A \cap T| \geqslant |\{u\}| = 1$. Hence we observe that $|S \cap B| = |A \cap T| = |\bar{A} \cap B| = 1$ and $A \cap B = \emptyset$ which implies that $A \cap \bar{B} \neq \emptyset$ because $|A| \geqslant 3$. Now we know that neither $\bar{A} \cap B$ nor $A \cap \bar{B}$ is empty. Thus, by Lemma 2.2(a), we observe that both $(\bar{A} \cap T) \cup (S \cap T) \cup (S \cap B)$ and $(A \cap T) \cup (S \cap T) \cup (S \cap \bar{B})$ are 5-cutsets of *G*. Since $1 = |\bar{A} \cap B| \leqslant |A \cap \bar{B}| < |A|$, and $y \in S$, this implies that either $\bar{A} \cap B$ or $A \cap \bar{B}$ is a fragment with respect to *xy* that is smaller than *A*. This contradiction completes the proof of Claim 3.1.1. $\square$

Claim 3.1.2. $|A \cap B| \neq 1$.

Proof. Assume that $|A \cap B| = 1$. Then we observe that $|S \cap B| = 1$ since $\bar{A} \cap B = \emptyset$ and $|B| = 2$. Since $xu \in E_G(x, A \cap T)$ and $xu \in E^{(2)}(G)$ we have $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| \geqslant 6$. Then by Lemma 2.2(a) we know that $\bar{A} \cap \bar{B}$ is empty. That also implies that $|(S \cap T) \cup (A \cap T)| \geqslant 6 - |S \cap B| = 5$. Hence $\bar{A} \cap T = T - ((S \cap T) \cup (A \cap T)) = \emptyset$. Since both $\bar{A} \cap B$ and $\bar{A} \cap \bar{B}$ are empty this means that $\bar{A} = \emptyset$ which contradicts the choice of A. Now Claim 3.1.2 is proved. $\square$

By Claims 3.1.1 and 3.1.2, we know that either $B \subset S$ or $B \subset A$. By Lemma 2.3 we know that $E_G(x, A) \cap E_L(G) = \emptyset$ which means $E_G(x, A) \subset E^{(2)}(G)$; hence if $B \subset A$, then $E_G(x, B) \subset E_G(x, A) \subset E^{(2)}(G)$. Thus, if $B \subset A$, then the conclusion of Proposition 3.1 follows from Lemma 3.3.

So we may assume that $B \subset S$ and $\bar{A} \cap B = A \cap B = \emptyset$ and $|S \cap B| = 2$. We show that $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| \leqslant 5$. Assume the contrary; then by Lemma 2.1(a) we know that $\bar{A} \cap \bar{B} = \emptyset$. Then, since both $\bar{A} \cap B$ and $\bar{A} \cap \bar{B}$ are empty, $|\bar{A}| = |\bar{A} \cap T| = |T| - |(S \cap T) \cup (A \cap T)| \leqslant 5 - 4 = 1$ which contradicts the choice of A.

We show that $|S \cap T| = 1$, $|A \cap T| = |S \cap \bar{B}| = 2$ and $A \cap \bar{B} \neq \emptyset$. From the above fact, we observe that $|A \cap T| = |(S \cap B) \cup (S \cap T) \cup (A \cap T)| - |(S \cap B) \cup (S \cap T)| \leqslant 5 - (2+1) = 2$ which implies that $A \cap \bar{B} \neq \emptyset$ since $A \cap B = \emptyset$ and $|A| \geqslant 3$. Hence, by Lemma 2.2(b) with $A, S, \bar{A}, B, T$ replaced by $\bar{B}, T, B, A, S$, respectively, we know that $|A \cap T| \geqslant |S \cap B| = 2$. We now know that $|A \cap T| = 2$. Moreover we observe that $|(S \cap B) \cup (S \cap T) \cup (A \cap T)| = 5$ and $|S \cap T| = 1$ which implies that $|S \cap \bar{B}| = 2$ since $|S| = 5$. Now it is shown that $|S \cap T| = 1$,$|A \cap T| = |S \cap \bar{B}| = 2$ and $A \cap \bar{B} \neq \emptyset$.

Next we show that $y \in B$. Since $|(A \cap T) \cup (S \cap T) \cup (S \cap \bar{B})| = 5$ we know that $A \cap \bar{B}$ is a fragment of G. This implies that $xy \notin E_G(x, T \cup \bar{B})$ or equivalently $y \in B$ since A is a minimum fragment with respect to $E(x) \cap E_L(G)$.

Since $A \cap \bar{B}$ is a fragment there is a vertex $u' \in N_G(x) \cap (A \cap \bar{B})$. Let B' be a minimum fragment with respect to xu'. Then $|B'| = 2$ since $xu' \in E_G(x, A) \subset E^{(2)}(G)$. Let $T' = N_G(B')$ and let $\bar{B}' = G - (T' \cup B')$. By Claims 3.1.1 and 3.1.2, we know that either $B' \subset S$ or $B' \subset A$.

We show that $B' \subset A$. Assume that $B' \subset S$. Then by the same argument as above, we know that $y \in B'$, say $B' = \{y, w\}$. Because $u' \in \bar{B}$ we know that $N_G(u') \cap B = \emptyset$ which implies that $B \neq B'$ and $yu' \notin E(G)$. Since $B \neq B'$ we know that $w \notin B$. Moreover $w \notin T$ because $S \cap T = \{x\}$ and $w \in B' \subset S$. Now we conclude that $w \in \bar{B}$ which implies that $yw \notin E(G)$. Hence $N_G(y) \subset B' \cup T' - \{y, u', w\}$. This means that $d_G(y) \leqslant |B' \cup T'| - |\{y, u', w\}| = 7 - 3 = 4$ which contradicts the choice of G. Now it is shown that $B' \subset A$.

The fact that $B' \subset A$ assures us that $E_G(x, B') \subset E_G(x, A) \subset E^{(2)}(G)$. Thus the conclusion of Proposition 3.1 follows from Lemma 3.3. Now the proof of Proposition 3.1 is completed. $\square$

4. Proofs of theorems

In this section we prove Theorems 1 and 2.

Proof of Theorem 1. Let G be a contraction critically 5-connected graph on n vertices and let $W(G) = \{x \in V(G) \mid E(x) \cap E^{(1)}(G) = \emptyset\}$. We use a very simple discharging process. Initially we assign to each vertex x of G the charge $|E(x) \cap E^{(1)}(G)|$. Then we discharge by the following rules:

(1) If either $x \in W(G)$ or $N_G(x) \cap W(G) = \emptyset$, then we move no charge from x.

(2) If neither $x \in W(G)$ nor $N_G(x) \cap W(G) = \emptyset$, then from x to each vertex of $N_G(x) \cap W(G)$, we move charge $(|E(x) \cap E^{(1)}(G)| - 1)/|N_G(x) \cap W(G)|$.

We let $\varphi(x)$ denote the amount of charge of x after the discharging. Then we know that $2|E^{(1)}(G)| = \sum_{x \in V(G)} |E(x) \cap E^{(1)}(G)| = \sum_{x \in V(G)} \varphi(x)$. By the discharging rule, we know that $\varphi(x) \geqslant 1$ for every vertex $x \in V(G) - W(G)$. Moreover, for each $x \in W(G)$, Proposition 3.1 assures us that there is a K_4^--configuration with center x. Hence, for each $x \in W(G)$ there are at least two vertices u and v in $N_G(x)$ such that $|E(u) \cap E^{(1)}(G)| = |E(v) \cap E^{(1)}(G)| = 4$ and $N_G(u) \cap W(G) = N_G(v) \cap W(G) = \{x\}$. From this fact we observe that $\varphi(x) \geqslant 3+3=6$ for any vertex $x \in W(G)$. Thus $2|E^{(1)}(G)| = \sum_{x \in V(G)} \varphi(x) \geqslant |V(G)| = n$. Now it is shown that the number of trivially noncontractible edges of G is greater than or equal to $n/2$ and the proof of Theorem 1 is completed. $\square$

Proof of Theorem 2. Let G be a contraction critically 5-connected graph on n vertices and let $U_6(G) = \{x \in V(G) \mid d_G(x) \geqslant 6\}$. Then Theorem C assures us that $U_6(G) \subset N_G(V_5(G))$. Hence $V(G) = V_5(G) \cup U_6(G) = V_5(G) \cup N_G(V_5(G))$. Let

$$\xi(G) = \sum_{z \in U_6(G)} (|E_G(z, V_5(G))| - 1).$$

Then we observe that $|E_G(U_6(G), V_5(G))| = |U_6(G)| + \xi(G)$.

Let H be the subgraph of G induced by the set of vertices of degree 5. Namely, $H = G[V_5(G)]$. Then, again by Theorem C, we know that $\delta(H) \geqslant 1$. Let $U_2(H) = \{x \in V_5(G) \mid d_H(x) \geqslant 2\}$. Then $V_5(G) = V(H) = V_1(H) \cup U_2(H)$. Let $W(H) = \{x \in V_5(G) \mid E(x) \cap E^{(1)}(G) = \emptyset\}$. We show that each vertex of $V_1(H)$ is incident with a trivially noncontractible edge, namely we show that $V_1(H) \cap W(H) = \emptyset$. For each $x \in W(H)$, Proposition 3.1 assures us that there is a K_4^--configuration with center x which means that $|N_G(x) \cap V_5(G)| \geqslant 2$. Thus we know that $W(H) \subset U_2(H)$ and $V_1(H) \cap W(H) = \emptyset$. Now it is shown that each vertex of $V_1(H)$ is incident with a trivially noncontractible edge.

Let x be a vertex of $V_1(H)$. Let xz be a trivially noncontractible edge which is incident with x. Then there is a vertex y in $V_5(G)$ such that $xy, yz \in E(G)$. If $z \in V_5(G)$, then $d_H(x) = |N_H(x)| \geqslant |\{y, z\}| = 2$ which contradicts the choice of x. Hence we know that $z \in U_6(G)$. Thus we know that for each $x \in V_1(H)$ there exists $z_x \in N_G(x) \cap U_6(G)$ such that $|N_G(z_x) \cap V_5(G)| \geqslant 2$. Let $Z = \{z_x \mid x \in V_1(H)\}$. For each $z \in Z$, take $x_z \in N_G(z) \cap V_5(G)$. If possible, we choose x_z so that $x_z \notin V_1(H)$. Set $X = \{x_z \mid z \in Z\}$. Then $|X \cap V_1(H)| \leqslant (1/2)|V_1(H)|$. We now observe that

$$\xi(G) \geqslant \sum_{z \in Z} |N_G(z) \cap V_5(G) - \{x_z\}| \geqslant |V_1(H) - X| \geqslant (1/2)|V_1(H)|.$$

Hence we have $|E_G(U_6(G), V_5(G))| = |U_6(G)| + \xi(G) \geqslant |U_6(G)| + (1/2)|V_1(H)|$.

On the other hand,

$$|E_G(U_6(G), V_5(G))| \leqslant \sum_{x \in V(H)} (d_G(x) - d_H(x)) \leqslant 4|V_1(H)| + 3|U_2(H)|.$$

Thus

$$\begin{aligned}|U_6(G)| &\leqslant |E_G(U_6(G), V_5(G))| - (1/2)|V_1(H)|\\ &\leqslant 4|V_1(H)| + 3|U_2(H)| - (1/2)|V_1(H)|\\ &= (7/2)|V_1(H)| + 3|U_2(H)|\\ &\leqslant (7/2)|V(H)| = (7/2)|V_5(G)|.\end{aligned}$$

Finally we have $n=|V(G)|=|U_6(G)|+|V_5(G)| \leqslant (9/2)|V_5(G)|$ and hence $(2n)/9 \leqslant |V_5(G)|$ which is the desired inequality and the proof of Theorem 2 is completed. □

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References

[1] K. Ando, K. Kawarabayashi, A. Kaneko, Vertices of degree 5 in a contraction critically 5-connected graphs, Graphs and Combinatorics, to be published.
[2] K. Ando, K. Kawarabayashi, A. Kaneko, Contractible edges in minimally *k*-connected graphs, SUT J. Math. 36 (1) (2000) 99–103.
[3] Y. Egawa, Contractible edges in *n*-connected graphs with minimum degree greater than or equal to $[\frac{5n}{4}]$, Graphs Combin. 7 (1991) 15–21.
[4] M. Fontet, Graphes 4-essentiels, C.R. Acad. Sci. Paris 287 (1978) 289–290.
[5] M. Kriesell, A degree sum condition for the existence of a contractible edge in a κ-connected graph, J. Combin. Theory Ser. B 82 (1) (2001) 81–101.
[6] W. Mader, Disjunkte Fragmente in kritisch *n*-fach zusammenhängende Graphen, European J. Combin. 6 (1985) 353–359.
[7] W. Mader, Generalizations of critical connectivity of graphs, Discrete Math. 72 (1988) 267–283.
[8] N. Martinov, Uncontractible 4-connected graphs, J. Graph Theory 6 (1982) 343–344.
[9] C. Thomassen, Non-separating cycles in *k*-connected graphs, J. Graph Theory 5 (1981) 351–354.
[10] W.T. Tutte, A theory of 3-connected graphs, Indag. Math. 23 (1961) 441–455.